Dreaming of a White (Dwarf) Christmas

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December 18, 2018

1 Background

A white dwarf is an interstellar object that marks the end of the life cycle of all except for the most massive stars in the universe. A white dwarf forms when all the star depletes all of the hydrogen in its core, and grows hot enough to begin fusing the helium in its core, and the hydrogen on the outer layers. Over time, the star will develop more layers, each with a distinct blend of carbon, oxygen, and helium. However, once a star has fully formed into a white dwarf, the effects of its chemical composition become less essential to understanding its structure. The final form of a white dwarf is essentially a degenerate electron gas with nuclei mixed in to balance the charge and create a gravitational attraction to hold the star together. This article discusses a derivation of the relationship between the mass and the radius of a white dwarf star. In this case, we are modeling the white dwarf as a uniform density sphere, and will be approximating $T = 0$, even though a white dwarf is actually extremely hot.

2 Deriving Gravitational Potential Energy

The first step to derive the relationship between the mass and the radius is to derive and understand the function for gravitational potential energy. Dimensional analysis can be used to argue that the gravitational potential energy of a uniform density sphere must be given by equation 1.

$$U_{grav} = -(constant)\frac{GM^2}{R} \qquad (1)$$

To prove the validity of this relationship, we first need to consider the units of gravitational potential energy, as well as the units of mass, radius, and the G constant. Gravitational potential energy has the standard unit of Joules $[J]$ or Newton-meters $[N*m]$. Mass is expressed in kilograms $[kg]$, radius is expressed in meters $[m]$, and the gravitational constant G has the units $[\frac{N*m^2}{kg^2}]$. Equation 2 shows how equation 1can be expressed as units. Looking at this equation, we can see that the gravitational constant cancels out the units of mass, and one of the units of the radius, leaving us with Newton-meters or Joules.

$$[N*m] = [\frac{N*m^2}{kg^2}][kg]^2[\frac{1}{m}] \qquad (2)$$

2.1 Why the Minus Sign?

One unusual feature of the function for gravitational potential energy is that the constant at the front is accompanied with a minus sign, which isn't typically seen when considering a value with units of energy. Gravitational potential energy is negative because it's opposing the work done by gravity, which is positive.

$$W_{grav} = (\frac{GM^2}{R^2})(R) = \frac{GM^2}{R} \qquad (3)$$

Equation 3 demonstrates that the work done by gravity is equal to the product of the gravitational force, and the distance R, which results in the same function as gravitational potential energy, only positive.

2.2 Deriving the Constant

As seen in equation 1, the function for gravitational potential energy has a constant in front of it, which we need to calculate. The constant can be derived by calculating the negative work needed to assemble the uniform density sphere shell by shell from the inside out. Figure 1 demonstrates this assembling process of the sphere, and the end result.

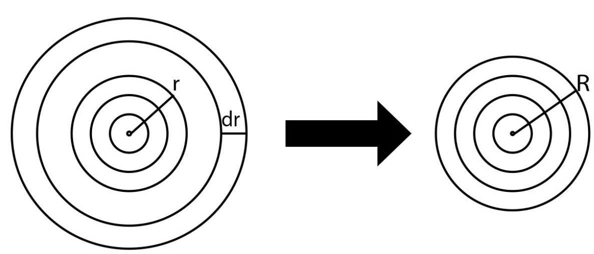

Figure 1: Assembling the Sphere Shell by Shell

The work needed to assemble one shell of the sphere is given by equation 4 Where G is the gravitational constant, m_1, is the mass of the sphere already assembled, and m_2 is the mass of the shell.

$$dU = -\frac{Gm_1m_2}{r} \tag{4}$$

Both the mass of sphere and the mass of the shell are given by the density times their volumes. The volume of the sphere is given by equation 5 where r is equal to the radius of the sphere shown in 1. The volume of the shell is given by the equation 6, which is the surface area of the shell times the thickness dr.

$$V_1 = \frac{4}{3}\pi r^3 \tag{5}$$

$$V_2 = 4\pi r^2 dr \tag{6}$$

Putting this together, the complete function for the work needed to assemble one shell is given by equation 7

$$dU = -\frac{G(\rho\frac{4}{3}\pi r^3)(\rho 4\pi r^2 dr)}{r} \tag{7}$$

To find the work needed to assemble the whole sphere, all that needs to be done is to take the integral of equation 7, the work needed to assemble a single shell. Equation 8 shows the integral function to find the total work.

$$U = \int_0^R -\frac{G(\rho\frac{4}{3}\pi r^3)(\rho 4\pi r^2 dr)}{r} \tag{8}$$

Equation 9 shows the result of taking this integral, and the completed function for gravitational potential energy:

$$U_{potential} = -\left(\frac{3}{5}\right)\frac{GM^2}{R} \tag{9}$$

As seen by this result, the constant in this function is $\frac{3}{5}$.

3 Deriving the Kinetic Energy

The next step in deriving the relationship between the mass and the radius is to find a function for the total kinetic energy of the degenerate electrons in the star. To do this, we start with equation 10, which represents the total energy of the electron gas as the energy of a single electron times the number of electrons.

$$U = \frac{3}{5}N\epsilon_f \tag{10}$$

The energy of a single electron is $\frac{3}{5}$ of the Fermi energy which is given by equation 11:

$$\epsilon_f = \frac{h^2}{8m}\left(\frac{3N}{\pi V}\right)^{\frac{2}{3}} \tag{11}$$

Plugging in equation 11 in for the Fermi energy in equation 10 gives us equation 12

$$U_{kinetic} = \frac{3}{5}N\left[\frac{h^2}{8m}\left(\frac{3N}{\pi V}\right)^{\frac{2}{3}}\right] \tag{12}$$

We know that the V is the total volume of the sphere, and that m is the mass of an electron. The only thing left to solve for is N, or the number of electrons. The number of electrons is the same as the number of protons in the star, so if we can solve for the number of protons, we will know the number of electrons. Since protons and neutrons are the particles accounting for the mass of the star, we can reason that the total mass of the star will be dependent on the mass of one proton, the number of protons, the mass of one neutron, and the number of neutrons. Additionally, since the number of protons is the same as the number of neutrons, and the mass of a proton is very close to the mass of a neutron, we can simplify the total mass as shown in equation 13

$$M = M_p + M_n = N_p m_p + N_n m_n = 2N_p m_p \tag{13}$$

From this we can solve for N_p as shown in equation 14, which is the same as N_e

$$N_e = \frac{M}{2m_p} \tag{14}$$

Plugging in all of these known values to equation 12, and using algebra to simplify the expression and condense all of the numerical constants gives us equation 15, the total kinetic energy of the electrons in the white dwarf.

$$U_{kinetic} = (0.0088)\left(\frac{h^2 M^{\frac{5}{3}}}{m_e m_p^{\frac{5}{3}} R^2}\right) \tag{15}$$

4 Finding the Equilibrium Radius as a Function of Mass

The equilibrium radius of the white dwarf is the radius that minimizes the total energy of the star. This makes the previous derivations for potential and kinetic energy useful, since knowing the total energy is an important step in finding the relationship between the mass and the radius of the white dwarf.

4.1 Total Energy as a Function of R

The total energy of the white dwarf is simply the sum of the gravitational potential energy and the kinetic energy. Equation 16 shows a function for the total energy, which is simply the sum of equation 9 and equation 15.

$$U_{tot} = -\left(\frac{3}{5}\right)\frac{GM^2}{R} + (0.0088)\left(\frac{h^2 M^{\frac{5}{3}}}{m_e m_p^{\frac{5}{3}} R^2}\right) \quad (16)$$

To get a basic idea of what this function looks like, we can think of it in a simpler form. This function of energy is essentially $-(constants)\frac{1}{R} + (constants)\frac{1}{R^2}$. Figure 2 shows a plot of what this basic function looks like.

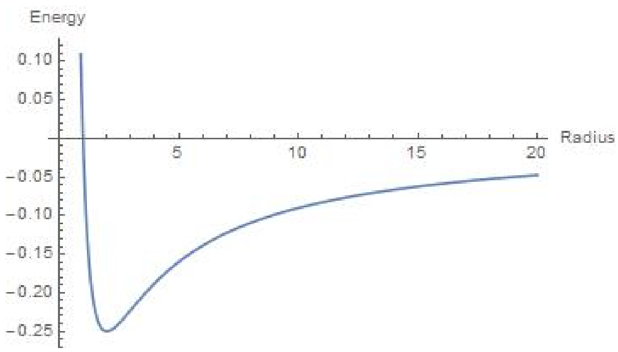

Figure 2: Plot of Total Energy vs. Radius

4.2 Equilibrium Radius as a Function of Mass

Finding the total energy as a function of radius was useful in finding the equilibrium radius because the point of equilibrium is that in which the total energy is minimized. Since this is a minimum point in the energy function, the first derivative at this point will be equal to zero. This means that the equilibrium radius can be found by taking the first derivative of equation 16, setting it equal to zero, and solving for R. Equation 17 shows the first derivative of the energy function:

$$\frac{dE}{dR} = \left(\frac{3}{5}\right)\frac{GM^2}{R^2} - (0.0088)\left(\frac{h^2 M^{\frac{5}{3}}}{m_e m_p^{\frac{5}{3}} R^3}\right) = 0 \quad (17)$$

The final step is to use algebra to rearrange equation 17 to solve for R. Equation 18 shows the final result for the equilibrium radius as a function of R.

$$R = (0.0088)\frac{10h^2 M^{\frac{-1}{3}}}{3m_e m_p^{\frac{5}{3}} G} \quad (18)$$

As shown by this formula for R in terms of M, the radius of the star decreases as the mass increases. Figure 3 shows a plot of equation 18, giving a visual picture of the relationship between the radius and mass of a white dwarf star. This graph shows an exponential decay in radius as the mass increases.

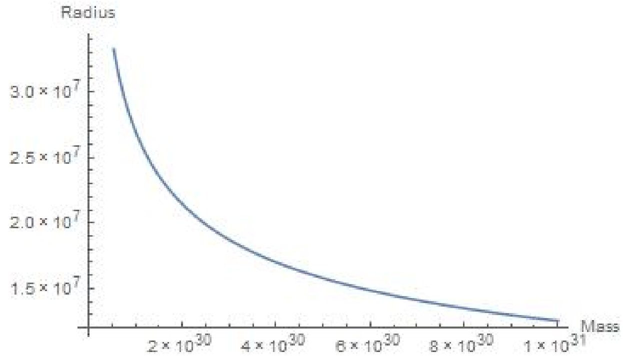

Figure 3: Plot of Radius vs. Mass

5 Evaluating the Equilibrium Radius of the Sun

To demonstrate how this function for the equilibrium radius can be applied to real life bodies, a quick calculation was done to compute the radius of the Sun. All of the values in this function are known since they're universal constants, so all that needs to be plugged in is the mass of the sun, which is $2 * 10^{30} kg$. Equation 19 shows the result of plugging in the mass of the Sun to equation 18 and evaluating.

$$R_{eq} = 7172.7km \quad (19)$$

6 Conclusions

Understanding the physics of white dwarfs is an essential part of understanding stellar evolution, since they are the last stage in the life cycle of most of the stars in the universe. The calculations done throughout this article demonstrate how we can easily derive a relationship between the mass and the equilibrium radius of a white dwarf star. The resulting function reveals an exponential decay in the equilibrium radius as the mass increases. These calculations also reveal a relationship between the equilibrium radius, the gravitational potential energy and the kinetic energy of the star. U_{grav} and $U_{kinetic}$ are in a constant battle with each other in a white dwarf star. U_{grav} makes the star want to collapse in on itself, while $U_{kinetic}$ makes the star want to expand. A white dwarf remaining within its equilibrium radius indicates a balance between the gravitational potential and kinetic energies of the star.